Week 3: Impedance vs mobility analogues, Q-factor

Microphone and Loudspeaker Design - Level 5

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What are we covering today?

1. Impedance analogy
2. Mobility analogy
3. Impedance characteristics
4. Q-factor
5. Tutorial questions

A weekly fact about Salford..!

Did you know...

- Salford is home to the first artificial canal in Britain! Opened in 1761, the Bridgewater Canal was the first artificial waterway fully independent of natural rivers. It was commissioned by Francis Egerton, 3rd Duke of Bridgewater, to transport coal from his mines in Worsley (posh part of Salford) to Manchester. It revolutionised the transportation system in England, and paved the way for the industrial revolution.

Impedance analogy

Impedance analogy

- For the **impedance analogy** we think of:
  - Force as being analogous to voltage $F \to V$
  - Velocity as being analogous to current $u \to I$
- By drawing this particular equivalence we preserve the analogy between mechanical and electrical impedance:

$$Z_M \to Z_E \tag{1}$$

- But, the topology of our problem is lost... i.e. mechanical system is arranged differently to its analogous electrical circuit
- Another popular one is called the mobility analogy...

Impedance analogy

Element	Impedance analogy	Mobility analogy
Mass	Mass $\leftrightarrow$ Inductor $Z_M = j\omega M_M \leftrightarrow Z_E = j\omega L_E$	
Spring	Spring $\leftrightarrow$ Capacitor $Z_M = \frac{1}{j\omega C_M} \leftrightarrow Z_E = \frac{1}{j\omega C_E}$	
Damper	Damper $\leftrightarrow$ Resistor $Z_M = R_M \leftrightarrow Z_E = R_E$	

Impedance analogy: mass-spring-damper

- Use analogy between mechanical and electrical components to model mechanical systems as electric circuits.
- To draw equivalent circuit first recall the definition of impedance analogy:

$$F \to V \qquad u \to I \tag{2}$$

- Note that the mass, spring and damper all have the same velocity, because they are connected together...

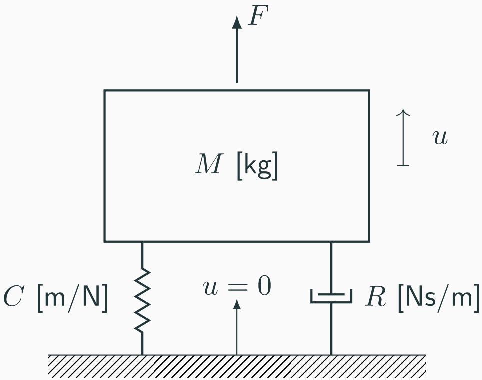


Figure 1: Mass-spring-damper.

Impedance analogy: mass-spring-damper

- Using AC circuit theory we can easily calculate the impedance of the mechanical system,

$$Z_M = j\omega M_M + \frac{1}{j\omega C_M} + R_M \tag{3}$$

- Mechanical velocity given by,

$$u = \frac{F}{Z_M} = \frac{F}{j\omega M_M + \frac{1}{j\omega C_M} + R_M} \tag{4}$$

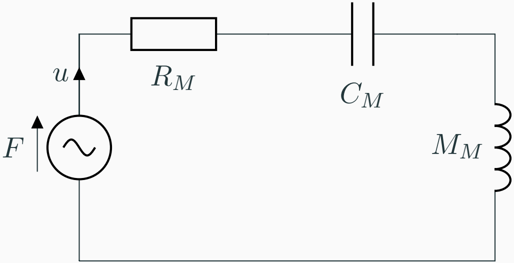


Figure 2: Mass-spring-damper equivalent circuit

Impedance analogy: mass-spring-damper

- Using equivalent circuit we can calculate the velocity of the mass,

$$u = \frac{F}{j\omega M_M + \frac{1}{j\omega C_M} + R_M} \qquad (5)$$

- As expected, the response looks just like an LCR circuit!

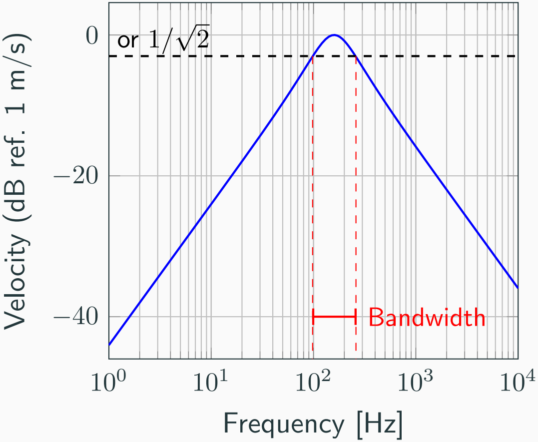


Figure 11: Velocity response of a mass-spring system.

Mobility analogy

Mobility analogy

- For the **impedance analogy** we made the following equivalences:
  - Force as being analogous to voltage $F \rightarrow V$ (drop parameter)
  - Velocity as being analogous to current $u \rightarrow I$ (flow parameter)
- But there is no reason why we cant consider the opposite!
- For the **mobility analogy** we make the following equivalences:
  - Force as being analogous to current $F \rightarrow I$ (flow parameter)
  - Velocity as being analogous to voltage $u \rightarrow V$ (drop parameter)

Mobility analogue: mass

- Mechanical impedance is

$$Z_M = \frac{F}{u} = j\omega M_M \tag{6}$$

- Mechanical mobility is

$$Y_M = \frac{u}{F} = \frac{1}{Z_M} = \frac{1}{j\omega M_M} \tag{7}$$

- According to the mobility analogy

$$\frac{u}{F} \rightarrow \frac{V}{I} \qquad \frac{1}{j\omega M_M} \rightarrow \frac{1}{j\omega C_E} \tag{8}$$

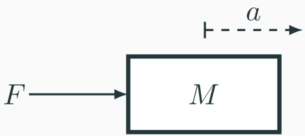


Figure 12: Mass element.

Mobility analogue: spring

- Mechanical impedance is

$$Z_M = \frac{F}{u} = \frac{1}{j\omega C_M} \tag{9}$$

- Mechanical mobility is

$$Y_M = \frac{u}{F} = \frac{1}{Z_M} = j\omega C_M \tag{10}$$

- According to the mobility analogy

$$\frac{u}{F} \to \frac{V}{I} \qquad j\omega C_M \to j\omega L_E \tag{11}$$

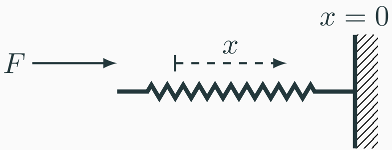


Figure 13: Spring element.

Mobility analogue: damper

- Mechanical impedance is

$$Z_M = \frac{F}{u} = R_M \tag{12}$$

- Mechanical mobility is

$$Y_M = \frac{u}{F} = \frac{1}{Z_M} = \frac{1}{R_M} \tag{13}$$

- According to the mobility analogy

$$\frac{u}{F} \to \frac{V}{I} \qquad \frac{1}{R_M} \to R_E \tag{14}$$

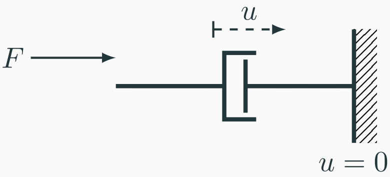


Figure 14: Damping element.

Impedance/mobility analogies: summary

Element	Impedance analogy	Mobility analogy
Mass	Mass $\leftrightarrow$ Inductor $Z_M = j\omega M_M \leftrightarrow Z_E = j\omega L$	Mass $\leftrightarrow$ Capacitor $\frac{1}{Z_M} = \frac{1}{j\omega M_M} \leftrightarrow Z_E = \frac{1}{j\omega C_E}$
Spring	Spring $\leftrightarrow$ Capacitor $Z_M = \frac{1}{j\omega C_M} \leftrightarrow Z_E = \frac{1}{j\omega C_E}$	Spring $\leftrightarrow$ Inductor $\frac{1}{Z_M} = j\omega C_M \leftrightarrow Z_E = j\omega L_E$
Damper	Damper $\leftrightarrow$ Resistor $Z_M = R_M \leftrightarrow Z_E = R_E$	Damper $\leftrightarrow$ Resistor $\frac{1}{Z_M} = \frac{1}{R_M} \leftrightarrow Z_E = R_E$

Mobility analogy: mass-spring-damper

- Use analogy between mechanical and electrical components to model mechanical systems as electric circuits.
- To draw equivalent circuit first recall the definition of mobility analogy:

$$F \to I \qquad u \to V \tag{15}$$

- Note that the mass, spring and damper all have the same velocity, because they are connected together...

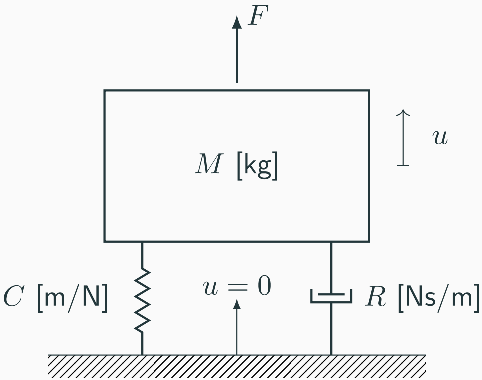


Figure 15: Mass-spring-damper.

Mobility analogy: mass-spring-damper

- Using AC circuit theory we can easily calculate the impedance of the mechanical system,

$$Z_E = \left(\frac{1}{j\omega C_M} + j\omega M_M + R_M\right)^{-1} \rightarrow Y_M = \frac{1}{Z_M} \tag{16}$$

- Mechanical velocity given by,

$$u = \frac{F}{Z_M} = \frac{F}{j\omega M_M + \frac{1}{j\omega C_M} + R_M} \tag{17}$$

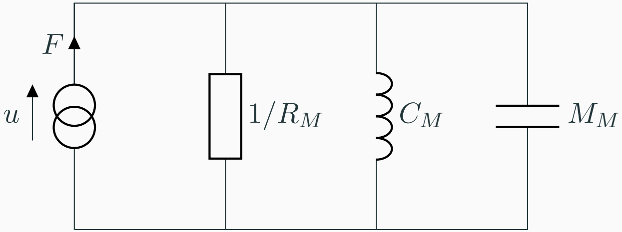


Figure 16: Mass-spring-damper equivalent circuit

Impedance vs. mobility analogy

- **Impedance analogy**
  - Retain the equivalence between impedance in the two domains
  - Topology of the circuit is not obvious (different layout to mechanical system)
- **Mobility analogy**
  - Loose the equivalence between impedance in the two domains
  - Topology of the circuit is the same as the layout of mechanical system

- Both circuits describe the same physical system (mass-spring-damper) but the roles or force and velocity are interchanged - *they are the 'dual' of one another*.
- Depending on the problem one may be more useful than the other...

Equivalent circuits: impedance vs. mobility

- Impedance:

$$F \to V \qquad u \to I \tag{18}$$

- Mobility:

$$F \to I \qquad u \to V \tag{19}$$

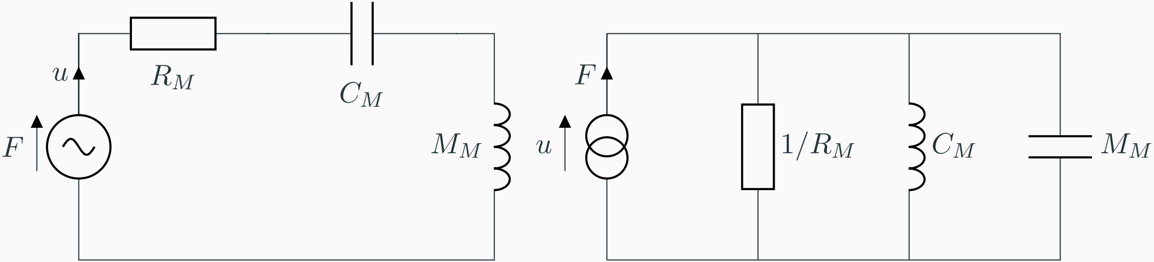


Figure 17: Equivalent circuits: impedance vs. mobility

Equivalent circuits: taking the dual

1. Replace inductors (with inductance L) with capacitors (with capacitance C) - and vice versa.
2. Replace resistors (with resistance R) with resistors of reciprocal resistance ($1/R$).
3. Replace the constant voltage sources with a constant current sources.
4. Parallel components become series and series components become parallel.

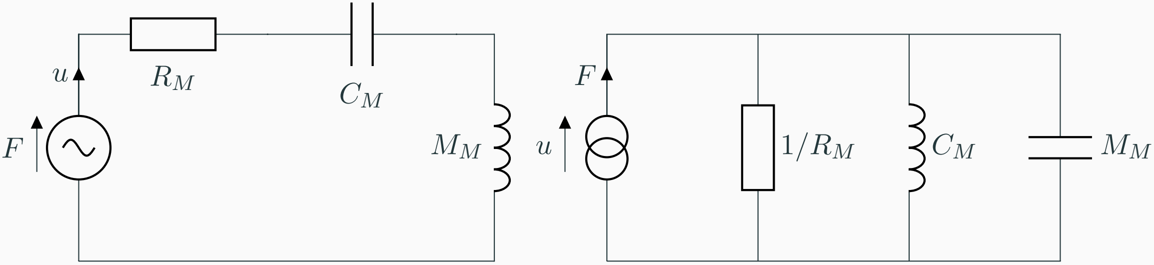


Figure 8: Equivalent circuits: impedance vs. mobility

Equivalent circuits: rules for taking the dual

- Have a go, take the dual of the following circuits:

Impedance characteristics

Impedance: a common language

- **Electrical impedance** is the measure of the opposition that a circuit presents to a current when a voltage is applied

$$Z_E = \frac{V}{I} \tag{20}$$

- **Mechanical impedance** is a measure of how much a structure resists motion (velocity) when subjected to a force

$$Z_M = \frac{F}{u} \tag{21}$$

- **Acoustic impedance** is a measure of the opposition that a system presents to the acoustic flow (volume velocity) when subjected to acoustic pressure

$$Z_A = \frac{p}{U} \tag{22}$$

Impedance: resistance vs. reactance

- Impedance is generally a **complex quantity**. It has a real part and an imaginary part.

$$Z = R + jX \tag{23}$$

- Real part is called the resistance - R - describes energy dissipation
- Imaginary part is called reactance - X - describes energy storage

- Mass-spring-damper example:

$$Z_M = j\omega M_M + \frac{1}{j\omega C_M} + R_M \quad \rightarrow \quad R_M + j\left(\omega M_M - \frac{1}{\omega C_M}\right) \tag{24}$$

Impedance: what does it look like

- Consider mechanical impedance of mass-spring-damper system

$$Z_M = R_M + j\left(\omega M_M - \frac{1}{\omega C_M}\right) \quad (25)$$

- When reactive parts are equal and opposite they cancel out - all that's left is the resistive part
- At resonance the impedance is a minimum

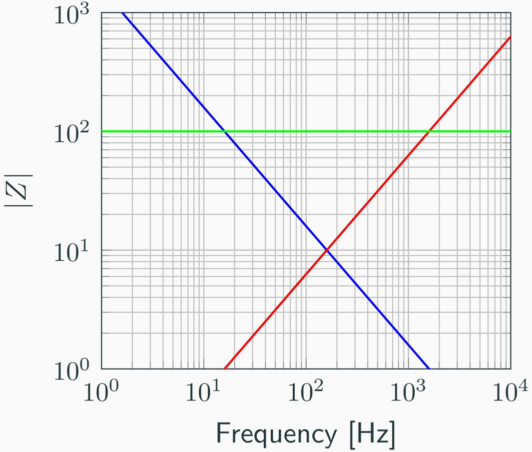


Figure 9: Impedance curves for mass, spring and damper.

Impedance: low damping

- Consider mechanical impedance of mass-spring-damper system

$$Z_M = R_M + j\left(\omega M_M - \frac{1}{\omega C_M}\right) \tag{26}$$

- Low damping

$$R_M << \omega_c M = \frac{1}{\omega_c C_M} \tag{27}$$

- Sharp notch at resonant frequency - minimum opposition to motion

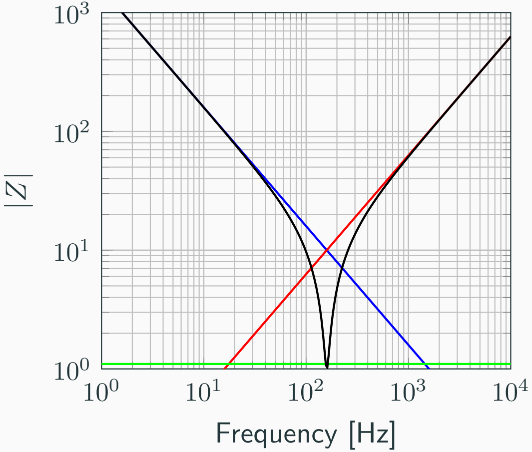


Figure 10: Impedance for low damping.

Impedance: high damping

- Consider mechanical impedance of mass-spring-damper system

$$Z_M = R_M + j\left(\omega M_M - \frac{1}{\omega C_M}\right) \tag{28}$$

- High damping

$$R_M >> \omega_c M = \frac{1}{\omega_c C_M} \tag{29}$$

- Broad minimum over range of frequencies

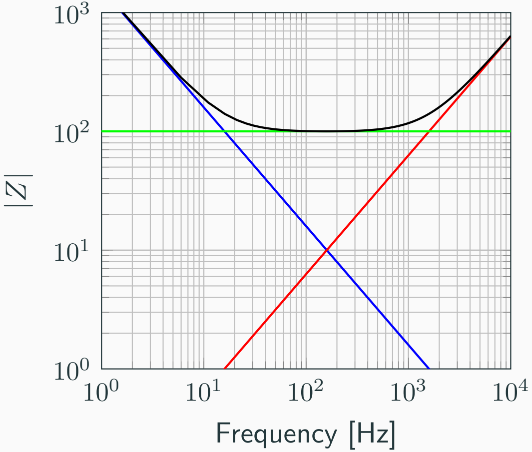


Figure 11: Impedance for high damping.

- The velocity is inversely proportional to impedance

$$u = \frac{F}{Z_M} \qquad (30)$$

- Larger damping (dashed plot) - lower velocity
- Lower damping (solid plot) - larger velocity

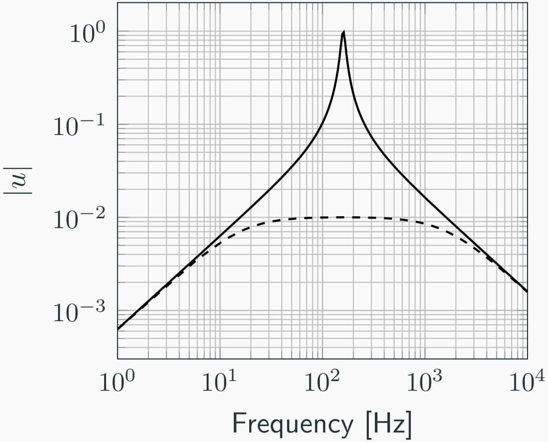


Figure 12: Velocity response.

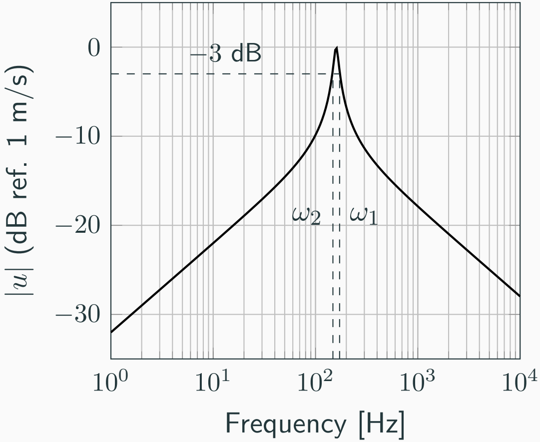
Q-factor

How do we quantify 'peakyness'?

- Damping clearly effects the sharpness of a resonant peak
- We quantify the 'peakyness' or sharpness by what we call the Q-factor

$$Q = \frac{\omega_c}{\Delta\omega} = \frac{\omega_c}{\omega_1 - \omega_2} \tag{31}$$

- $\Delta\omega$ is the full half power bandwidth - i.e. when $|u|^2 = |u_{max}|^2/2$ or $|u| = |u_{max}|/\sqrt{2}$



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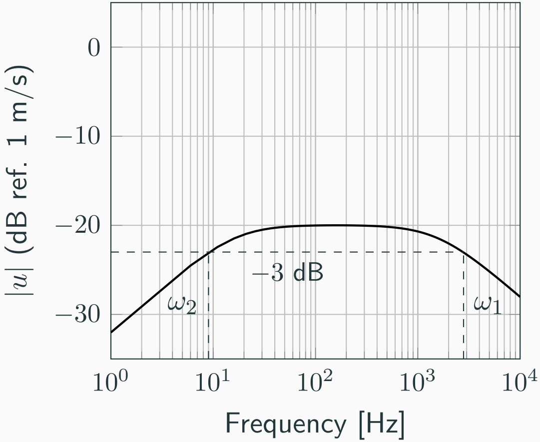

Figure 14: Velocity response.

How do we quantify 'peakyness'?

- We want an equation for Q in terms of mechanical (or electrical/acoustical) component values - 3 steps:

1. Find resonant frequency ω_c
2. Find frequencies that have half power
3. Plug into definition!

$$Q = \frac{\omega_c}{\Delta\omega} = \frac{\omega_c}{\omega_1 - \omega_2} \tag{33}$$

4. Do a little algebra...

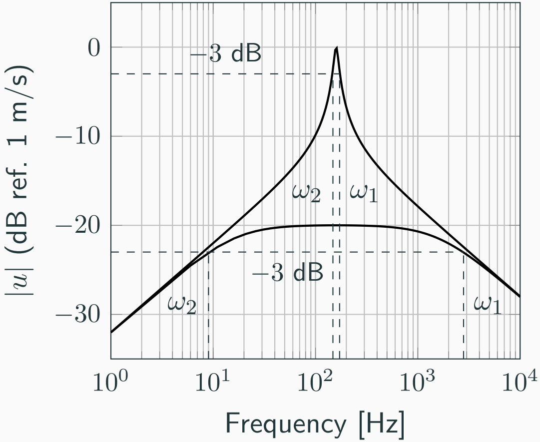

Figure 15: Velocity response.

- Definition of Q factor

$$Q = \frac{\omega_c}{\Delta\omega} = \frac{\omega_c}{\omega_1 - \omega_2} \qquad (34)$$

- In terms of component values

$$Q = \frac{M_M}{R_M}\sqrt{\frac{1}{M_M C_M}} = \frac{1}{R_M}\sqrt{\frac{M_M}{C_M}} \qquad (35)$$

- Explore limits
  - As $R \to \infty$, $Q \to 0$
  - As $R \to 0$, $Q \to \infty$

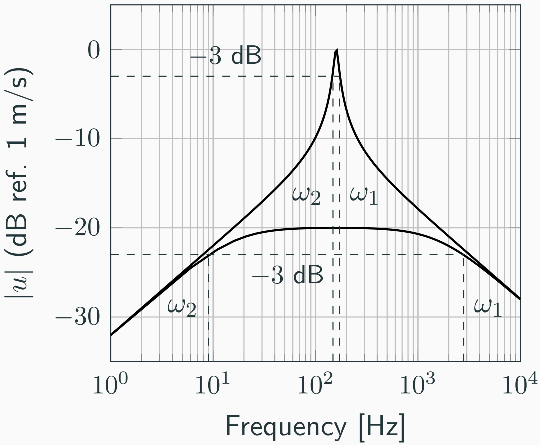

Figure 15: Velocity response.

How do we quantify 'peakyness'?

- Q factor is also measure of how far the impedance at resonance is from a point reactive impedance

$$Q = \frac{\omega_c M_M}{R} = \frac{Z_{MM}}{Z_{MR}} \qquad (36)$$

- Q factor can be used to define three regimes
  - $Q > \frac{1}{2}$ under-damped
  - $Q < \frac{1}{2}$ over-damped
  - $Q = \frac{1}{2}$ critically-damped

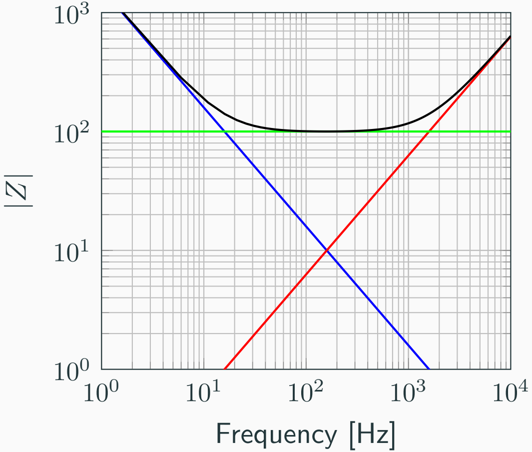

Figure 16: Impedance for high damping.

Next week...

- Equations of motion - transient and stready state analysis
- Acoustic domain

- Reading:
  - Mechanical domain: lecture notes, chp. 4, pg. all